Exercise 2.5: fractal boundaries

In the book *Understanding Nonlinear Dynamics* by D. Kaplan and L. Glass (Springer, New York, 1995), the authors propose the study of the following map:

\[ x_{t+1} = (x_t + a + b \sin 2\pi x_t) \pmod{1} \]  

(1)

Here, the “Modulus” operator takes the fractional part of a number. For instance, the modulus of 4.28, that is \( 4.28 \pmod{1} \) is 0.28. In R, the modulus operator is written as \%. It may be interesting to see the effect of the modulus operator on the map (1). The following lines plot the map with and without the modulus:

\[
\begin{align*}
\text{f.x<- function(x,a,b,pi2)\{ (x+a+b*sin(pi2*x))\%\%1 \} \# (mod 1) } \\
\text{f.x1<- function(x,a,b,pi2)\{ (x+a+b*sin(pi2*x)) \} \# without modulus} \\
\text{### parameters and initial conditions} \\
\text{a<- 0.53} \\
\text{b<- 0.62} \\
\text{xinit1<- 0.0} \\
\text{pi2<- 2*pi} \\
\text{plot(0,0,type="n",xlim=c(0,4.5),ylim=c(0,4.5),xlab="x(t)",ylab="x(t+1)",cex.lab=1.5,cex.axis=1.2)} \\
\text{curve(f.x1(x,a,b,pi2),from=0,to=4.5,lty=1,lwd=2,col="black",add=T)} \\
\text{curve(f.x(x,a,b,pi2), from=0,to=4.5,lty=2,lwd=2,col="blue",add=T)}
\end{align*}
\]

The result is reported in Fig. 1. We see that without (mod 1) the map (1) is a periodic increasing function (solid line), while with the modulus an almost stationary periodic behaviour is apparent (dashed line). Note that in the next Figura 1 Plot of the map (1) \((a = 0.53\) and \(b = 0.62\)) with (dashed line) and without (solid line) the modulus operator.

Fig. 2, only segment of the (1) (with the modulus operator) are plotted (dashed lines) in the interval \([0,1]\).

**Exercise:** Study the map starting from different initial conditions.

**Solution:** We use the following code, in which \(a = 0.53\) and \(b = 0.62\), as in D. Kaplan and L. Glass. The initial condition \(x_0\) (in the code \(xinit\)) is equal to 0.06 and 0.07.

\[
\begin{align*}
\text{# Basin1: cobweb plots with two initial conditions} \\
\text{f.x<- function(x,a,b,pi2)\{ (x+a+b*sin(pi2*x))\%\%1 \} } \\
\text{pi2<- 2*pi} \\
\text{##### function to draw the cobweb plot #####} \\
\text{iter<- function(xinit,nstep,a,b,pi2){ \# starting function iter} 
\end{align*}
\]
The result is shown in Fig. 2. We are in presence of two stable periodic cycles and so we speak of multistability. If \( x_{\text{init}} = 0.06 \) there is a 2-period cycle, with fixed points \( x_t = 0.6773 \) and \( x_{t+1} = 0.650867 \). If \( x_{\text{init}} = 0.07 \) there is a 4-period cycle, with fixed points \( x_t = 0.39848, x_{t+1} = 0.29765, x_{t+2} = 0.42007, x_{t+3} = 0.24852 \).

Let \( x_0 \) be in the whole interval \([0, 1]\). There are only two possible behaviors for the map: the iterates either approach the 2-period cycles, or approach
Figura 2 Cobweb plot of the map (1) with \(a = 0.53\) and \(b = 0.62\). With the initial condition \(x_{\text{init}} = 0.06\), the iterates converge to a 2-period cycle (dot-dash line), while with \(x_{\text{init}} = 0.07\), the convergence is toward a 4-period cycles (solid line). The (1) is plotted with dashed lines.

The 4-period cycles. With the following code, a black vertical segment is plotted when the iterates converge to the 2-period cycle, while a gray vertical segment is plotted when the iterates converge to the 4-period cycle. Our intention is to distinguish the two basins of attraction, that is to identify the boundaries of each basin. The following code teaches us that the task is not so clear-cut.

```r
#Basin2: fractal boundaries
f.x<-function(x,a,b,pi2){(x+a+b*sin(pi2*x))%%1}
pi2<-2*pi
a<- 0.53
b<- 0.62
ntrans<- 500  # transient
n<-50  # number of iterations after the transient
nt<-ntrans+n  # total iterations
nr<-1000  # number of initial conditions in [0,1]
xinit<- 0
xfin<- 1
pa<-(xfin-xinit)/nr  # length of the step from 0 to 1
xinit<- xinit-pa
yv<- 0.5  # length of vertical segments
par(mfrow=c(3,1))  # create a multi-paneled plot with 3 rows and 1 column.
plot(1,1,type="n",xlim=c(0,1),ylim=c(0,yv),xlab="",ylab="",yaxt="n",
cex.lab=1.8,cex.axis=1.5)
```

3
for(x in 1:nr) {
  xinit<-xinit+pa
  x<-xinit
  for (j in 1:nt) {
    y<- f.x(x,a,b,pi2)
    if(j>ntrans){
      if((x>0.297) & (x<0.298)) segments(xinit,0,xinit,yv,lwd=1,col="gray")
      if((x>0.650) & (x<0.651)) segments(xinit,0,xinit,yv,lwd=1,col="black")
      x<-y
    }
  }
}

############################ first zooming-in

xinit<- 0.
xfin <- 0.1
nr<-5000
pa<-(xfin-xinit)/nr
xinit<-xinit-pa
yv<- 0.5
plot(1,1,type="n",xlim=c(xinit,xfin),ylim=c(0,yv),xlab="",ylab="",yaxt="n",
cex.lab=1.8,cex.axis=1.5)
for(x in 1:nr) {
  xinit<-xinit+pa
  x<-xinit
  for (j in 1:nt) {
    y<-f.x(x,a,b,pi2)
    if(j>ntrans){
      if((x>0.297) & (x<0.298)) segments(xinit,0,xinit,yv,lwd=1,col="gray")
      if((x>0.650) & (x<0.651)) segments(xinit,0,xinit,yv,lwd=1,col="black")
      x<-y
    }
  }
}

############################ second zooming-in

xinit<- 0.06
xfin <- 0.08
pa<-(xfin-xinit)/nr
xinit<-xinit-pa
yv<- 0.5
plot(1,1,type="n",xlim=c(xinit,xfin),ylim=c(0,yv),xlab="x(t)",ylab="",yaxt="n",
cex.lab=1.8,cex.axis=1.5)
for(x in 1:nr) {
  xinit<-xinit+pa

4
\begin{verbatim}
x<-xinit
for (j in 1:nt) {
  y<- f.x(x,a,b,pi2)
  if(j>ntrans){
    if((x>0.297) & (x<0.298))segments(xinit,0,xinit,yv,lwd=1,col="gray")
    if((x>0.650) & (x<0.651))segments(xinit,0,xinit,yv,lwd=1,col="black")
  }
  x<-y
}
\end{verbatim}

Figure 3 shows that when we start with $x_0 \in [0, 1]$ and we see that there are large enough regions in which two nearby initial conditions lead to the same fixed point. But if we zoom in on the small interval $x_0 \in [0, 0.1]$ , we see that the region $[0, 0.1]$ which appeared almost entirely black contains in fact smaller gray zones. The same result happens if we zoom in on the even smaller region $[0.06, 0.08]$. The conclusion is that we cannot trace a line delimiting the boundaries of the basins, but they are interleaved in a fractal way.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{On the axes of abscissae are reported the intervals of the initial conditions from which of the iterates of the map (1) ($a = 0.53$ and $b = 0.62$) start. Top: [0, 1], middle: [0, 0.1], bottom: [0.06, 0.08]. A black vertical segment is plotted when the iterates converge to the 2-period cycle and a gray vertical segment is plotted when the iterates converge to the 4-period cycle.}
\end{figure}